Construction of Chaotically Synchronizing Systems and Some Methods of Controlling Chaos

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We propose a method of contructing chaotically synchronizing systems and give examples of two control algorithms based on properties of Lipschitz maps. Bifurcation diagrams show the robustness of the method for broad band of control parameters.

Perhaps the most well known of the properties that characterize chaotic systems is their sensitivity to initial conditions. Two identical chaotic systems that are started from virtually identical initial conditions in a short time would be observed to diverge from one another.

However, for chaotic systems one observes a very important phenomenon called chaotic synchronization. This is a nonlinear phenomenon such that the behaviors of a subsystem are chaotic but take the same values as those in another subsystem. Pecora and Caroll [1] proposed a method for synthesizing a nonlinear system with chaotic synchronization in a so called master-slave configuration. The P–C method is very simple and applied to many chaotic systems [2, 3]. Chaotic synchronization is a very useful phenomenon in engineering and many secure communication methods using it are proposed [4, 5]. There, an information signal containing a message is transmitted using a chaotic signal as a broadband carrier, and the synchronization is necessary to recover the information at the receiver. For example in [6, 7] the information signal is added to the chaotic signal and in [5] a parametric modulation is used for the transmission of digital signals.

In this paper we propose a new method for constructing chaotically synchronizing systems. The method can also be used as a control methodology for stabilizing a fixed point or a periodic orbit embedded in a chaotic attractor. The method is based on properties of Lipschitz maps.

Let $x(n+1) = F(x(n))$ be a discrete-time dynamical system, i.e. $x(n) \in R^n$ is the state of the system at time n, and F is a mapping from R^n to R^n.

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Definition

Let $x(n+1) = F(x(n))$ and $y(n+1) = G(y(n))$ be two discrete-time dynamical systems in R^n. Let the solutions of the systems be given by $x(n; n_0, x_0)$ and $y(n; n_0, y_0)$, respectively. We say that F *synchronizes in-phase* with G if there exists a subset of R^n denoted by $D(n_0)$ such that $x_0, y_0 \in D(n_0)$ implies

$$\| x(n; n_0, x_0) - y(n; n_0, y_0) \| \to 0 \quad \text{as} \quad n \to \infty . \qquad (1)$$

The synchronization is defined *global* if $D(n_0)$ spans the whole space, i.e. $D(n_0) = R^n$. It is defined as *local* if $D(n_0)$ is a proper subset of R^n. We can call $D(n_0)$ the *region of synchronization.*

We say that F *synchronizes anti-phase* with G if under the same conditions as above

$$\| x(n; n_0, x_0) + y(n; n_0, y_0) \| \to 0 \quad \text{as} \quad n \to \infty . \qquad (2)$$

Definition

Let X be a normed space. A map $f: X \to X$ is called L-Lipschitz if there exists $L \in R^1$ such that for any $x, y \in X$

$$\| f(x) - f(y) \| \le L \| x - y \| .$$

A map f is called contraction if it is L-Lipschitz and $L < 1$.

Proposition

If $x(n+1) = F(x(n))$ is a discrete-time dynamical system on R^n, then for each contraction mapping K there exists a discrete-time dynamical system G on R^n: $y(n+1) = G(y(n))$ which synchronizes globally with F.

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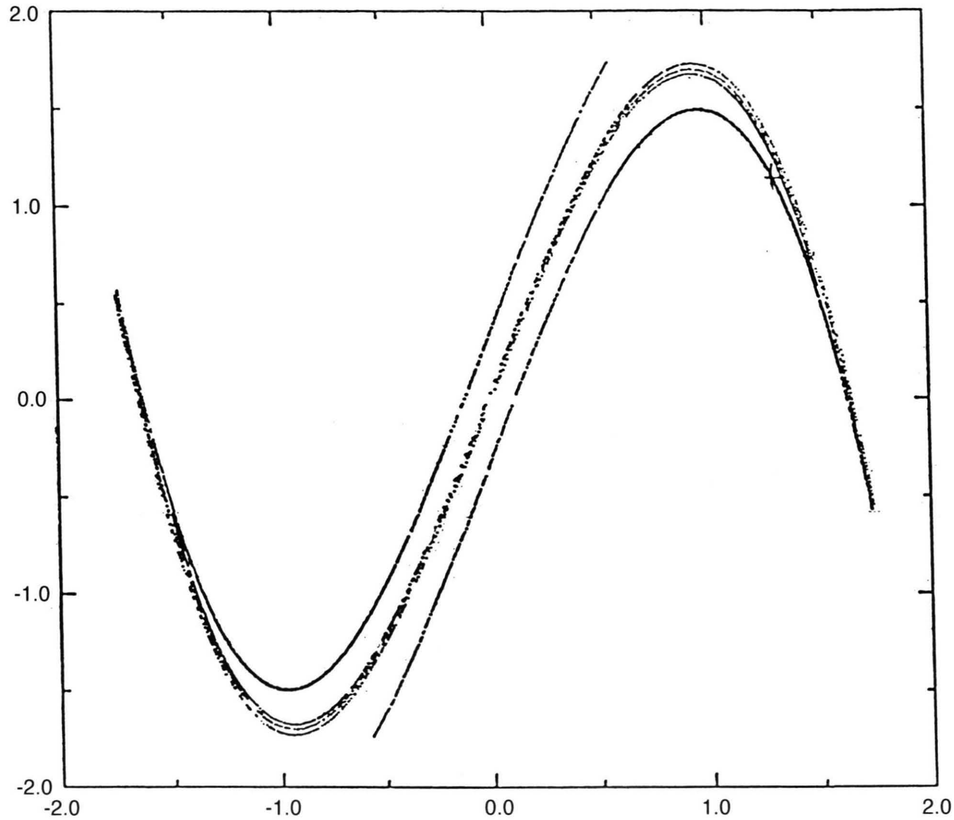

Fig. 1. Chaotic attractor of the Holmes map.

Proof:

Every map $F\colon R^n \to R^n$ can be rewritten as

$F = K + F - K$, where $K\colon R^n \to R^n$, K-contraction.

Now the system G is constructed as

$$y(n+1) = G(y(n)) \equiv K(y(n)) + (F - K)(x(n)),$$
$$x(n+1) = F(x(n)).$$

F and G are in-phase synchronized on R^n:

$$\| x(n+1) - y(n+1) \| = \| F(x(n)) - G(y(n)) \| = \| K(x(n) + (F-K)(x(n)) - K(y(n)) - (F-K)(x(n)) \| = \| K(x(n)) - K(y(n)) \| \le \alpha_K \| x(n) - y(n) \|.$$

Because $\alpha_K < 1$, we have

$$\| x(n+1) - y(n+1) \| \to 0 \quad \text{as} \quad n \to \infty.$$

As an example we take the Holmes map. The map was introduced by Holmes [8] as a simple model of a Poincaré section for the Duffing system. The map is described by

$$x(n+1) = y(n),$$
$$y(n+1) = a\,x(n) - x^3(n) - b\,y(n),$$
$$x(n),\ y(n),\ a,\ b \in R^1.$$

For a = 2.75 and b = 0.18 the system has a chaotic attractor (see Figure 1).

Let us define

$$K(x, y) = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then $(F - K)(x, y) = \begin{pmatrix} 0.5\,y \\ (a - 0.5)\,x - x^3 - b\,y \end{pmatrix}$.

K is a contraction mapping because $\| K \| < \frac{1}{2}$. The new system

$$x_1(n+1) + 0.5\,y_1(n) + 0.5\,y(n)$$
$$y_1(n+1) = 0.5\,x_1(n) + (a - 0.5)\,x(n) - x^3(n) - b\,y(n)$$

synchronizes with the (x, y) system (see Figure 2).

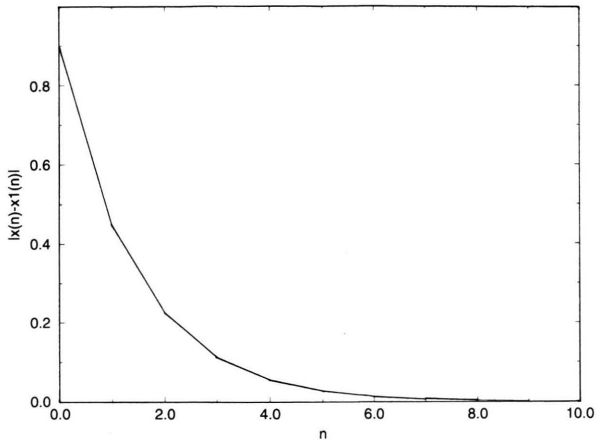

Fig. 2. The difference $|x(n) - x_1(n)|$ for the Holmes map.

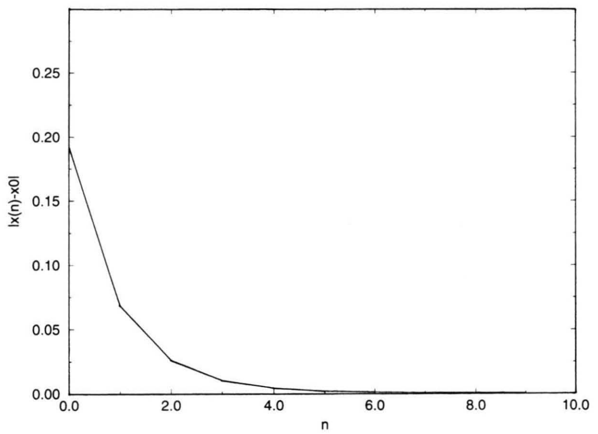

Fig. 3. The difference $|x(n) - x_0|$ for the logistic map.

Now let us describe a method of stabilization of a chaotic trajectory onto period-1 fixed point.

Let $x(n+1) = F(x(n))$ be a discrete-time dynamical system on R^n with an unstable fixed point x_0; $F(x_0) = x_0$.

Proposition

For every contraction mapping K on R^n there exists a control $C: R^n \to R^n$ such that the system

$$x(n+1) = F(x(n)) + C(x(n))$$

converges as $n \to \infty$ to the fixed point x_0.

Proof:

Let us define

$$C(x(n)) = K(x(n)) - F(x(n)) - K(x_0) + F(x_0).$$

Then

$$x(n+1) = K(x(n)) - K(x_0) + x_0,$$
$$\| x(n+1) - x_0 \| \le \alpha_K \| x(n) - x_0 \| \quad \text{so}$$
$$x(n+1) \to x_0 \quad \text{as} \quad n \to \infty.$$

The above approach can be readily extended to control the system into higher, period-k fixed points. It suffice to consider the system with a new map

$$G = F^k = \underbrace{F \cdot F \cdot \ldots \cdot F}_{k}$$

instead of the map F in the control scheme. By doing so, the control is activated every k mapping steps.

Example

As an example we consider the logistic map

$$x(n+1) = C - x^2(n), \quad \text{where} \quad x(n) \in [-2, 2] \quad \text{and} \quad C = 1.94.$$

The map has an unstable fixed point $x_0 = 0.98$.

Let us define $K(x(n)) = \frac{1}{5} x^2(n)$.

Then the control $C(x(n)) = \frac{6}{5} x^2(n) - \frac{6}{5} x_0^2$ and the controlled system has the form

$$x(n+1) = C + \tfrac{1}{5} x^2(n) - \tfrac{6}{5} x_0^2.$$

We have

$$\| x(n+1) - x_0 \| = \| \tfrac{1}{5} x^2(n) - \tfrac{1}{5} x_0^2 \|$$
$$\le \tfrac{1}{5} \| x(n) - x_0 \| \, \| x(n) + x_0 \| \le \tfrac{4}{5} \| x(n) - x_0 \|.$$

Hence $x(n) \to x_0$ as $n \to \infty$ (see Figure 3).

Now we describe an another method of control of a fixed point of a discrete-time dynamical system. The method is based on the properties of Lipschitz maps.

Theorem

Suppose that t_n lies in [0, 1], that Σt_n is divergent and that F is L-Lipschitz and $\limsup t_n < \dfrac{2}{L+1}$.

Then the iteration

$$x(n+1) = (1 - t_n) x(n) + t_n F(x(n))$$

converges to a fixed point of F for each initial point $x(0)$.

Let us define the control scheme for a discrete-time dynamical system $x(n+1) = F(x(n))$ with F L-Lipschitz.

$$x(n+1) = F(x(n)) + \delta_{n,p}\,\varepsilon(x(n) - F(x(n)), \quad (3)$$

where $\delta_{n,p} = 1$ when n is a multiple of $p \in \mathrm{N}$ and zero otherwise, and $\varepsilon \geq \frac{L-1}{L+1}$. So we see that every p time steps our system is modified by means of a feedback $\varepsilon(x(n) - F(x(n)))$. The condition of the Theorem are fulfilled, so the iteration (3) converges to a fixed point of F for each initial condition $x(0)$.

Conclusion

The given examples of constructing chaotically synchronizing systems and of control schemes always bring the system to the desired form without, e.g., the paradoxical problem of multistability. Moreover, the synchronization and control can work on global scale in phase space, i.e. it does not depend on the initial state of the process. The approach developed in this paper works in general for n-dimensional maps and can also be applied to the control of flows on the Poincaré section.

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